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TEMPERATURE CALCULATION IN AN INCOMPRESSIBLE PERMANENT LAMINAR FLUID FLOW THROUGH A CIRCULAR PIPE WITH AXIAL CONDUCTION AND VISCOSITY

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Abstract—A solution for the energy equation, with a parabolic velocity profile, for an incompressible permanent laminar fluid flow through a circular pipe, including viscous dissipation and axial conduction is shown. The solution is presented as a series of hypergeometric functions. Furthermore, it is shown how to apply this solution to problems involving first, second, and third kind boundary conditions. A first kind boundary condition example is examined, for Péclet numbers from 5 to 100.

NOMENCLATURE

a	pipe radius
Br	Brinkman number, $\mu U^2/KT_0$
c	fluid specific heat
K	thermal conductivity
l	non-dimensional heat transfer coefficient
$M_{a,b}$	Whittaker function of order a, b
Pe	Péclet number, $\rho Uac/K$
r	radial coordinate
r^+	non-dimensional radial coordinate
$R_m(r^+), Z_n(z^+), \Phi(\phi)$	functions whose product is the homogeneous solution of equation (2), for n (integer) > 0
$R_{mn}(r^+)$	solution of the energy equation for $m \geqslant 1$
T^+	temperature function, T/T_0
U	fluid velocity at the pipe axis
x	variable, $(r^+)^2$
X_{mn}	function that is the product of r^+ by $R_{mn}(r^+)$
z	axial coordinate
z^+	non-dimensional axial coordinate, $z/a\,Pe$
Z_{mn}	series term yielding $Z_n = 1+\sum_{m=1}^{\infty} Z_{mn}$.

Greek symbols

λ_{mn}	constant such that $Z_{mn} = e^{-\lambda^2\, mnz^+}$
μ	viscosity coefficient
ρ	fluid density
ϕ	angular coordinate
ω_{mn}	variable, $\lambda_{mn}(1+\lambda_{mn}^2/Pe^2)/4$.

1. INTRODUCTION

THE HEAT transfer problem in a circular pipe has been extensively studied by Reynolds [1, 2], Sparrow and Lin [3], and other authors. All solutions for the energy equation have been found in terms of a series of functions, using the separation of variables method. Ash and Heinbockel [4] established a solution as a series of hypergeometric functions of the radial coordinate, assuming a constant wall temperature. Faghri and Welty [5] published a rather comprehensive paper, where a solution was obtained using a radial coordinate power series expansion. They applied their solution to a second kind boundary condition problem.

This paper represents an extension to Faghri and Welty's solution, where the solution is obtained as a radial coordinate Whittaker functions series. Thus, it is now possible to treat first, second, and third kind boundary condition problems in a closed way, as shown in Section 3. It is also possible to study the effect of low Péclet numbers, a situation that could occur in the slow flow of high-temperature liquid metals. This situation may be important in nuclear reactors, if the working fluid is slowed due to a pump malfunction.

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2. SOLUTION OF THE ENERGY EQUATION

The energy equation for a parabolic velocity profile in a circular pipe, under permanent, laminar, and incompressible fluid flow regime, in cylindrical coordinates, is given by Welty *et al.* [6]

$$\rho cU\left[1-\left(\frac{r}{a}\right)^2\right]\frac{\partial T}{\partial z} = K\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}\right\}+\frac{4\mu U r^2}{a^4}, \quad (1)$$

where ρ is the fluid density, c the specific heat, k the thermal conductivity, μ the viscosity coefficient, U the fluid velocity on the pipe axis, a the pipe radius, and T the temperature function. It is also assumed that ρ, c, K, and μ are all constants.

Transforming equation (1) into a non-dimensional form, with

$$r^+ = \frac{r}{a}, \quad z^+ = \frac{z}{a\,Pe}, \quad T^+ = \frac{T}{T_0},$$

where $Pe = ecUa/K$ is the Péclet number, and T_0 a reference arbitrary temperature, the non-dimensional energy equation becomes

$$(1-r^{+2})\frac{\partial T^+}{\partial z^+} = \frac{1}{r^+}\frac{\partial}{\partial r^+}\left(r^+\frac{\partial T^+}{\partial r^+}\right) + \frac{1}{r^{+2}}\frac{\partial^2 T^+}{\partial \phi^2} + \frac{1}{Pe^2}\frac{\partial^2 T^+}{z^{+2}} + 4Br\, r^{+2}, \quad (2)$$

where $Br = \mu U^2/KT_0$ is the Brinkman number.

In order to solve the above non-homogeneous partial differential equation, a particular solution for the complete equation is added to the general homogeneous equation solution. For convenience, the particular solution is assumed to be of the form

$$H(r^+, z^+) = Cz^+ + f(r^+) + H_0, \quad (3)$$

where H_0 and C are constants to be determined.

The use of $H(r^+, z^+)$ splits the original equation into two other equations, one of them being the energy equation without the viscous effects.

Substituting this function into the energy equation results in the following solution being obtained

$$H(r^+, z^+) = Cz^+ + C\left(\frac{r^{+2}}{4} - \frac{r^{+16}}{16}\right) - \frac{Br}{4} r^{+4} + H_0. \quad (4)$$

The homogeneous equation is solved by the method of separation of variables, using

$$T^+_{\text{hom}}(r^+, \phi, z^+) = \psi(r^+, z^+)\Phi(\phi). \quad (5)$$

This relation yields the solution, for n (integer) > 0

$$\Phi_n(\phi) = A_n \cos(n\phi) + B_n \sin(n\phi) \quad (6)$$

and

$$\psi_n(r^+, z^+) = R_n(r^+)Z_n(z^+). \quad (7)$$

Hence, the energy equation becomes

$$(1-r^{+2})R_n\frac{dZ_n}{dz^+} = \frac{Z_n}{r^+}\frac{d}{dn^+}\left(r^+\frac{d}{dr^+}R_n\right) - \frac{n^2}{r^{+2}}R_nZ_n + \frac{R_m}{Pe^2}\frac{d^2Z_n}{dz^{+2}}, \quad (8)$$

which is the same energy equation without the viscous effects.

In order to solve equation (8) by the method of separation of variables, Z_n is expressed in terms of the series for the integer m

$$Z_n = 1 + \sum_{m=1}^{\infty} Z_{mn}, \quad (9)$$

where

$$Z_{0n} = 1 = \text{const.} \quad \text{and} \quad Z_{mn} = e^{-\lambda^2 mn^{z^+}}.$$

This assures that

$$\frac{1}{Z_n}\cdot\frac{dZ_n}{dz^+} = -\lambda_n^2 = \text{const.}$$

and also

$$\frac{1}{Z_n}\cdot\frac{d^2Z_n}{dz^{+2}} = \lambda_n^4,$$

and therefore the separation of variables is possible. The solution of equation (8), for $m = 0$, is

$$R_{0n} = D_{0n}r^{+n} + E_{0n}r^{+(-n)}. \quad (10)$$

Since R_{0n} must be finite for $0 \leqslant r^+ \leqslant 1$. E_{0n} must be zero.

For $m \geqslant 1$ the energy equation becomes

$$-\lambda_{mn}^2(1-r^{+2})R_{mn} = \frac{1}{r^+}\frac{d}{dr^+}\left(r^+\frac{dR_{mn}}{dr^+}\right) - \frac{n^2}{r^{+2}}R_{mn} + \frac{\lambda_{mn}^4}{Pe^2}R_{mn}. \quad (11)$$

Faghri and Welty [5] solved this equation through a series expansion, and applied it to a second kind boundary condition problem.

A closed form solution can be obtained after some transformations, as seen below. Introducing the variable $x = r^{+2}$, and the function $X_{mn}(x) = r^+R_{mn}(r^+)$, equation (11) becomes

$$-\lambda_{mn}^2(1-x)x^{-1/2}X_{mn} = x^{-3/2}X_{mn} + 4x^{1/2}\frac{d^2x_{mn}}{dx^2} - \frac{n^2}{x^2}x^{-1/2}X_{mn} + \frac{\lambda_{mn}^4}{Pe^2}x^{-1/2}X_{mn}. \quad (12)$$

Rearranging the terms, and equating them to zero

$$\frac{d^2X_{mn}}{dx^2} + \left\{\frac{\lambda_{mn}^2}{4} + \frac{\lambda_{mn}^2}{4x}\left(1 + \frac{\lambda_{mn}^2}{Pe^2}\right) + \frac{1-n^2}{4x^2}\right\}X_{mn} = 0. \quad (13)$$

The solution of this equation is

$$X_{mn} = F_{mn}M_{\omega_{mn},n/2}(\lambda_{mn}x) + G_{mn}W_{\omega_{mn},n/2}(\lambda_{mn}x), \quad (14)$$

where $\omega_{mn} = \lambda_{mn}/4(1+\lambda_{mn}^2/Pe^2)$, and the functions $M_{\omega_{mn},n/2}(\lambda_{mn}x)$ and $W_{\omega_{mn},n/2}(\lambda_{mn}x)$ are the known Whittaker functions [6]. The function $W_{\omega_{mn},n/2}(\lambda_{mn}x)$ becomes infinite at the origin for all ω_{mn} and for all $n \geqslant 0$. In order to have $R_{mn}(r^+)$ finite for $0 \leqslant x \leqslant 1$, G_{mn} must be zero. Then the general solution for the energy equation is

$$T^+ = Cz^+ + C\left(\frac{r^{+2}}{4} - \frac{r^{+4}}{16}\right) - \frac{Br}{4}r^{+4} + H_0 + \sum_{n=0}^{\infty}[A_n\cos(n\phi) + B_n\sin(n\phi)]r^{+n} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{l^{-\lambda^2 nm^{z^+}}}{r^+}[C_{mn}\cos(n\phi) + D_{mn}\sin(n\phi)]M_{\omega_{mn},n/2}(\lambda_{mn}r^{+2}). \quad (15)$$

The constants in equation (15) are determined from the boundary conditions.

3. APPLICATION TO BOUNDARY CONDITIONS

As an example, for first kind boundary conditions, independent of z^+, it is assumed that

$$T^+(r^+, \phi, 0) = h_1(r^+, \phi), \quad (16)$$

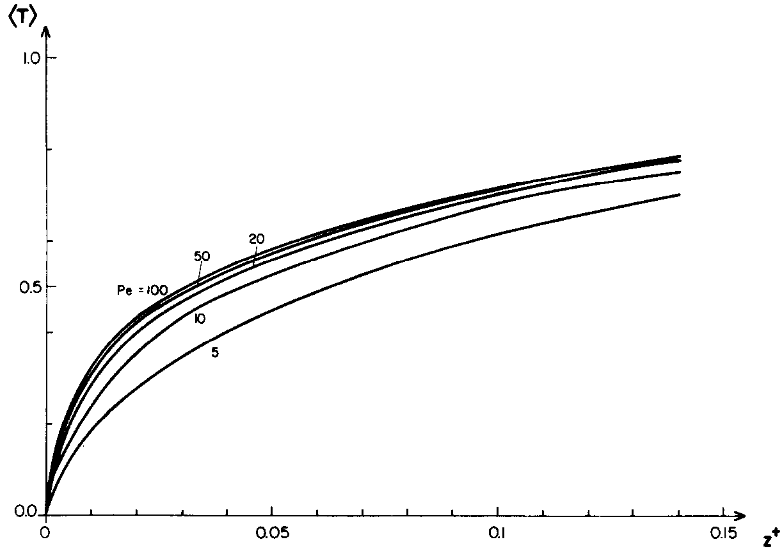

FIG. 1. Variation of $\langle T \rangle$, the average fluid temperature, with z^+ and Pe.

$$T^+(1, \phi, z^+) = h_2(\phi). \qquad (17)$$

Under these conditions, $C = 0$, and

$$M_{\omega_{mn}, n/2}(\lambda_{mn} r^{+2}) = 0. \qquad (18)$$

Therefore, equation (18) determines the λ_{mn} values. The other coefficients are determined by

$$A_n = H^c_{2,m}, \qquad (19)$$

and

$$B_n = H^s_{2,m}, \qquad (20)$$

where $H^c_{2,n}$ and $H^s_{2,n}$ are cos $(n\phi)$ and sin $(n\phi)$ Fourier series expansion coefficients of $h_2(\phi)$, respectively.

Finally, equation (16) yields C_{mn} and D_{mn}. Usually the C_{mn} and D_{mn} computation is done by looking at the minimum quadratic deviation in the solution for $z^+ = 0$ [5].

For second kind boundary conditions

$$T^+(r^+, \phi, 0) = h_1(r^+, \phi), \qquad (21)$$

$$\left.\frac{dT^+}{dr^+}\right|_{r^+=1} = q(\phi), \qquad (22)$$

where $q(\phi)$ is the non-dimensional heat flux.

Condition (22) determines the values of λ_{mn} through the equation

$$\frac{d}{dr^+}\left[\frac{1}{r^+} M_{\omega_{mn}, n/2}(\lambda_{mn} r^{+2})\right]_{r^+=1} = 0, \qquad (23)$$

and also the constants C, A_n and B_n through the identification of a Fourier series. Finally, condition (21) determines the other coefficients of equation (15).

After computing the values of λ_{mn}, the problem is similar to a first kind boundary condition one.

Finally, for third kind boundary conditions

$$T^+(r, \phi, 0) = h_1(r^+, \phi), \qquad (24)$$

$$\left.\frac{\partial T^+}{\partial r^+}\right|_{r^+=1} = l[T^+(r^+ = 1) - T_{ref}(\phi)], \qquad (25)$$

where $T_{ref}(\phi)$ is a non-dimensional reference temperature, and l is the non-dimensional heat transmission coefficient. From equation (25), C must be zero. Using equation (25) for $z^+ \to \infty$, the constants A_n and B_n are determined after expanding $T_{ref}(\phi)$ into a Fourier series. From equations (8) and (9) for z^+ finite

$$2_{mn}\frac{d}{dy} M_{\omega_{mn}, n/2}(y)|_{y=\lambda_{mn}x} = (l+1) M_{\omega_{mn}, n/2}(\lambda_{mn} x), \qquad (26)$$

is obtained, and the calculation of C_{mn} and D_{mn} is similar to that of the previous cases.

Numerical calculations were performed for the first kind boundary conditions below

$$T^+(1, \phi, z^+) = 1 + \cos\phi, \qquad (27)$$

$$T(r^+, \psi, 0) = 0, \qquad (28)$$

and Pe from 5 to 100. The results are shown in Figs. 1 and 2. The variations of $\langle T \rangle$, the average temperature, and Nu, the Nusselt number, as a function of z^+ and Pe, are presented in Figs. 1 and 2.

4. CONCLUSIONS

From the numerical computation of the first kind boundary conditions example, a few conclusions can be drawn. For $Pe = 5$, the case of a slow flow of a high temperature liquid metal, one can infer that λ_{mn} increases with n, which implies that the variation in the wall temperature, with respect to the angular variable, is transmitted in a more efficient way than the mean value of the wall temperature. For larger Pe, the opposite is true. Normally, Pe is larger than 100;

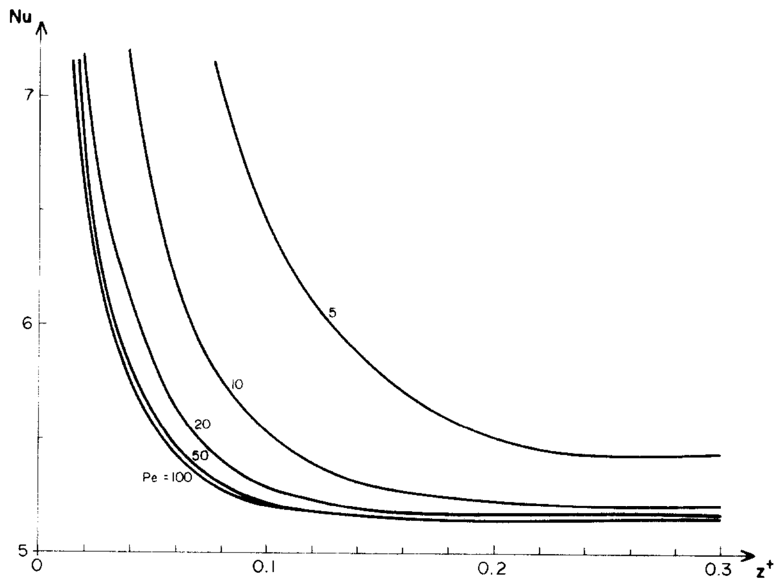

FIG. 2. Variation of Nu, the Nusselt number, with z^+ and Pe.

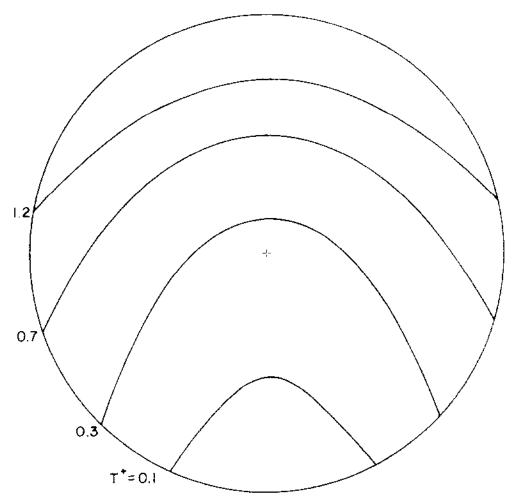

FIG. 3(a). Temperature profile for $z^+ = 0.1$, for $Pe = 5$.

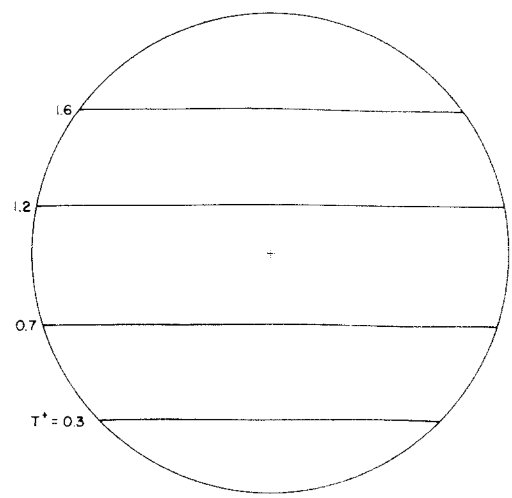

FIG. 3(c). Temperature profile for $z^+ = 1.0$, for $Pe = 5$.

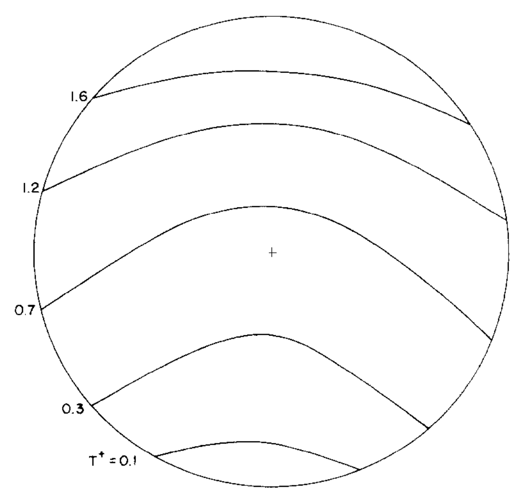

FIG. 3(b). Temperature profile for $z^+ = 0.2$, for $Pe = 5$.

however, if a high-temperature liquid metal, such as sodium, is flowing slowly Pe can be as low as 5. It has to be pointed out that, when Pe is smaller than 100, its effect on $\langle T \rangle$ and Nu has to be taken into account in designing heat transfer systems. Table 1 presents the first ten values of λ_{mn} for $Pe = 5$, $n = 0$ and 1. Figures 3(a)–(c) show the temperature profiles for $Pe = 5$ on three cross-sections of the pipe.

Numerically, the computation of λ_{mn} when Pe is low (5–20) presents some problems. This is due to the fact that the Whittaker functions grow very fast with increasing arguments. This difficulty is overcome by decomposing the Whittaker functions into a product of two functions [7], one of them being a confluent hypergeometric function of the type ${}_1F_1$, thus making it possible to compute the λ_{mn}'s.

Table 1. Values of λ_{mn} for $Pe = 5$

m	$n = 0$	$n = 1$
1	2.3853	3.5314
2	4.5109	5.2700
3	5.9765	6.5791
4	7.1579	7.6729
5	8.1743	8.6315
6	9.0798	9.4950
7	9.9039	10.2870
8	10.6653	11.0226
9	11.3763	11.7124
10	12.0456	12.3640

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REFERENCES

1. W. C. Reynolds, Heat transfer to fully developed flow in a circular tube with arbitrary circumferential heat flux, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **82**, 108–112 (1960).
2. W. C. Reynolds, Turbulent heat transfer in a circular tube with variable circumferential heat flux, *Int. J. Heat Mass Transfer* **6**, 445–454 (1963).
3. E. M. Sparrow and S. H. Lin, Turbulent heat transfer in a tube with circumferentially varying temperature or heat flux, *Int. J. Heat Mass Transfer* **6**, 866–876 (1963).
4. R. L. Ash and J. H. Heinbockel, Note on heat transfer in laminar, fully developed pipe flow with axial conduction, *Z. Angew. Math. Phys.* **21**, 266–269 (1970).
5. M. Faghri and J. R. Welty, Analysis of heat transfer, including axial fluid conduction, for laminar tube flow with arbitrary circumferential wall heat flux variation, *Int. J. Heat Mass Transfer* **21**, 317–323 (1978).
6. J. R. Welty, C. E. Wicks and R. E. Wilson, *Fundamentals of Momentum, Heat and Mass Transfer*. Wiley, New York (1969).
7. H. Buchholz, *Confluent Hypergeometric Function*. Springer, Berlin (1969).

CALCUL DE LA TEMPERATURE DANS UN ECOULEMENT INCOMPRESSIBLE PERMANENT A L'INTERIEUR D'UNE CONDUITE CIRCULAIRE AVEC CONDUCTION AXIALE ET VISCOSITE

Résumé—Une solution de l'équation de l'énergie, avec profil de vitesse parabolique, est donnée pour un écoulement incompressible permanent dans une conduite circulaire avec dissipation visqueuse et conduction axiale. La solution est présentée sous forme d'une série de fonctions hypergéométriques. De plus, on montre comment appliquer cette solution aux problèmes comportant des conditions aux limites de première, seconde et troisième espèces. Un exemple de conditions aux limites de première espèce étudié pour des nombres de Péclet de 5 à 100.

BERECHNUNG DES TEMPERATURFELDES EINER INKOMPRESSIBLEN, STATIONÄREN, LAMINAREN FLÜSSIGKEITSSTRÖMUNG IN EINEM KREISROHR MIT AXIALER WÄRMELEITUNG UND VISKOSITÄTS-EFFEKTEN

Zusammenfassung—Für inkompressible, stationäre, laminare Flüssigkeitsströmung mit parabolischem Geschwindigkeitsprofil im Kreisrohr wird eine Lösung der Energiegleichung angegeben, wobei reibungsbedingte Dissipation und axiale Wärmeleitung berücksichtigt werden. Die Lösung wird in Form von Reihen hypergeometrischer Funktion dargestellt. Weiter wird die Anwendung dieser Lösung auf Probleme mit Randbedingungen erster, zweiter und dritter Art gezeigt. Für Peclet-Zahlen von 5 bis 100 wird ein Beispiel mit der Randbedingung erster Art untersucht.

РАСЧЕТ ТЕМПЕРАТУРЫ ЛАМИНАРНОГО ПОТОКА НЕСЖИМАЕМОЙ ЖИДКОСТИ В КРУГЛОЙ ТРУБЕ С АКСИАЛЬНОЙ ТЕПЛОПРОВОДНОСТЬЮ И ВЯЗКОСТЬЮ

Аннотация—Получено решение уравнения энергии с параболическим профилем скорости для случая ламинарного течения несжимаемой жидкости в круглой трубе с учетом вязкой диссипации и аксиальной теплопроводности. Решение представлено в виде ряда гипергеометрических функций. Кроме того показано, как такое решение может применяться в задачах с граничными условиями первого, второго и третьего рода. Рассмотрен пример с граничным условием первого рода при значениях числа Пекле от 5 до 100.